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Solutions to the Variational Equations for Relative Motion of Satellites

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The use of integrals of certain reference satellite motions has played an important role in many classical investigations. As a principal example, the solution to the two-body problem is the foundation of Lagrange's planetary equations. The two-body solution has also been used to find solutions that satisfy the associated linear differential "variational" equations that form the basis of certain guidance schemes without additional integration by expanding the solution about a nominal trajectory. These guidance equations have traditionally been written in a nonrotating coordinate system. Recently, solutions to the two-body (and the perturbed two-body) variational equations written in a rotating coordinate system have been found by linearizing solutions to nonlinear equations of relative motion obtained from the two two-body problem solutions. In those solutions, classical and modified orbital elements are varied. Here, the general problem of finding solutions to the variational equations of dynamical systems that are completely integrable is first considered. Then, an alternate solution for linear relative motion of a satellite with respect to another satellite moving in an elliptic orbit is obtained in the form of an analytical state transition matrix by varying the initial polar coordinates, the inclination, and the right ascension of the ascending node of the reference orbit in the analytical solution to the nonlinear equations. Only the inverse of a relatively simple matrix is required at initial time and the new transition matrix is valid for arbitrary elliptic orbits. Examples of relative motion, obtained by evaluating the analytical solution, are presented and are compared with numerical results.

Introduction

THE "variational equations" of nonlinear dynamic systems play fundamental roles in the analysis and control of such systems. These equations are obtained by linearizing the governing nonlinear differential equations about a particular, or "reference" solution. Often the particular solution is constant, that is, it is an equilibrium point, and the variational equations are time-invariant. In such a case, the solutions to the variational equations provide information on the stability of the motion relative to that point. When the variational equations are time-invariant, they can also be used very effectively in the development of feedback control laws. Even when the reference solution is time-varying and the variational equations, generally, have time-varying coefficients, they can be used effectively to test for stability and to devise guidance and control laws. When the nonlinear system is autonomous and the reference solution is constant, then the variational equations are linear differential equations with constant coefficients that may be solved using standard methods. On the other hand, when the reference solution is time-varying, the variational equations generally have time-varying coefficients and are much more difficult to solve.

The two-body problem is an example of a nonlinear dynamic system for which there are several different types of particular solutions. If polar coordinates are used, linearizing the equations of the two-body problem about the particular solution corresponding to a circular orbit results in variational equations that have constant coefficients (see, for example, [1], p. 455 and [2]) and are essentially the same as the Hill–Clohessy–Wiltshire (HCW) equations [3,4]. In the case of a reference particular solution that corresponds to an elliptic orbit, the two-body variational equations have periodic coefficients and, as discussed further later in this paper, are very difficult to solve. Broucke [5] provides a list of the many authors who have obtained exact and/or approximate solutions to the equations.

In this paper, we first consider the general problem of finding the solutions to variational equations of completely integrable, autonomous systems. Then, we consider the problem of finding a general, exact solution to the linear variational equations for relative motion with respect to an elliptic orbit (reference orbit) in a rotating coordinate system. The "new" solution is obtained by varying the initial polar coordinates, the inclination, and longitude of the ascending node of the reference orbit. We use the descriptor new because the solution should give the same results as others, but the variables are different and the construction of the transition matrix given here does not require the inverse of a very complicated matrix, only a simple one. For a circular reference orbit the equations reduce to the HCW equations. Results from the solution are compared with corresponding results obtained by numerically integrating linear differential equations, nonlinear equations from the two-body equation of motion, and results obtained using the HCW transition matrix.

Variational Equations

As noted in the introduction, the variational equations play several important roles in the analysis of the dynamics of nonlinear systems and the development of control laws and guidance algorithms. Here, we focus on nonlinear, autonomous systems and do not consider control variables. Let such a system be described by a state vector $\boldsymbol{X} = [X_1 \quad X_2 \quad X_3 \quad \dots \quad X_n]^T$ and the differential system,

$$\dot{\boldsymbol{X}} = \mathbf{F}(\boldsymbol{X}), \qquad \boldsymbol{X}(t_0) = \boldsymbol{X}_0 \tag{1}$$

where t_0 is the initial time. The variational equations obtained by expanding the nonlinear, autonomous, differential Eqs. (1) about a particular solution $\boldsymbol{X}_p$ are

$$\dot{\boldsymbol{x}} = \{[\partial\mathbf{F}(\boldsymbol{X})/\partial\boldsymbol{X}]|_{\boldsymbol{X}=\boldsymbol{X}_p}\}\boldsymbol{x}, \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{2}$$

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where $\boldsymbol{x} = \boldsymbol{X} - \boldsymbol{X}_p$. Equation (2) may be rewritten in the familiar form,

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x}, \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{3}$$

where $\mathbf{A} = \{[\partial \mathbf{F}(\boldsymbol{X})/\partial \boldsymbol{X}]|_{X=X_P}\}$. The solution to Eq. (3), in terms of the state transition matrix $\boldsymbol{\Phi}(t, t_0)$ is

$$\boldsymbol{x} = \boldsymbol{\Phi}(t, t_0)\boldsymbol{x}_0 \tag{4}$$

The state transition matrix $\boldsymbol{\Phi}(t, t_0)$ is the solution to

$$\dot{\boldsymbol{\Phi}} = \mathbf{A}\boldsymbol{\Phi} \tag{5}$$

that satisfies the initial condition

$$\boldsymbol{\Phi}(t_0, t_0) = \mathbf{I}_{n\times n} \tag{6}$$

where $\mathbf{I}_{n\times n}$ is the $n \times n$ identity matrix.

If the particular solution $\boldsymbol{X}_p$ is an equilibrium point and Eqs. (3) are time-invariant, then Eqs. (3) are easy to solve. On the other hand, if $\boldsymbol{X}_p$ is a time-varying particular solution, then, generally, the matrix $\mathbf{A}$ is time-varying and the equations are more difficult to solve. However, when a general solution to the nonlinear Eqs. (1) is known, it contains all the particular solutions and the information on how the solution changes, or varies, as initial conditions and/or parameters are "varied." Thus, when a general solution to Eqs. (1) is available, one is guaranteed that an analytical solution to Eqs. (3) exists.

To make this more evident, let the general solution to Eqs. (1) be written as

$$\boldsymbol{X} = \left[\, g_1(\boldsymbol{X}_0, t_0, t) \quad g_2(\boldsymbol{X}_0, t_0, t) \quad \ldots \quad g_n(\boldsymbol{X}_0, t_0, t)\,\right]^T \tag{7}$$

where the g_j, $j = 1, 2, 3, \ldots, n$ are functional forms of the initial conditions, $\boldsymbol{X}_{j0}$ $j = 1, 2, 3, \ldots, n$, the initial time t_0, and the time t. Then, it is clear that $\frac{\partial X_j}{\partial t} = \frac{\partial g_j}{\partial t}$, $j = 1, 2, 3, \ldots, n$ satisfy Eqs. (1). Moreover, assuming that all the functions in the general solution are continuous, with continuous first and second derivatives [1],

$$\frac{\partial(\delta X_j)}{\partial t} = \frac{\partial \delta g_j(\boldsymbol{X}_0, t_0, t)}{\partial t}, \qquad j = 1, 2, 3, \ldots, n \tag{8}$$

where

$$\delta X_j(t) = \sum_{k=1}^{n} \frac{\partial g_j(\boldsymbol{X}_0, t_0, t)}{\partial X_k(t_0)} \delta X_k(t_0), \qquad j = 1, 2, 3, \ldots, n \tag{9}$$

and the $\delta X_k(t_0)$, $k = 1, 2, 3, \ldots, n$, are arbitrary constants. We can set $\boldsymbol{x} = [\,\delta X_1 \quad \delta X_2 \quad \delta X_3 \quad \ldots \quad \delta X_n\,]^T$

It is then apparent that a particular choice of $\boldsymbol{X}_0$ at $t = t_0$ generates the particular solution in Eqs. (1). Thus, the state transition matrix for the variational system of equations is

$$\boldsymbol{\Phi}(t, t_0) = \begin{bmatrix} \frac{\partial g_1}{\partial \boldsymbol{X}_0} \\ \frac{\partial g_2}{\partial \boldsymbol{X}_0} \\ \vdots \\ \frac{\partial g_n}{\partial \boldsymbol{X}_0} \end{bmatrix} \tag{10}$$

and the solution is

$$\boldsymbol{x} = \boldsymbol{\Phi}(t, t_0)\boldsymbol{x}_0 \tag{11}$$

Most integrable, autonomous, nonlinear differential systems have solutions in which the time is expressed in terms of one of the states, or as in the two-body problem, a variable (the eccentric anomaly) related to one of the states (the true anomaly). Thus, we have

$$\boldsymbol{X} = \left[\, g_1(\boldsymbol{X}_0, \xi_0, \xi) \quad g_2(\boldsymbol{X}_0, \xi_0, \xi) \quad \ldots \quad g_n(\boldsymbol{X}_0, \xi_0, \xi)\,\right]^T \tag{12}$$

where ξ is obtained from an equation of the form,

$$t - t_0 = f(\boldsymbol{X}_0, \xi_0, \xi) \tag{13}$$

When the solution is of the form of Eqs. (12) and an equation of the form of Eqs. (13) is part of the nonlinear general solution,

$$\begin{aligned} \frac{\partial(X_j)}{\partial \boldsymbol{X}_0} &= \left(\frac{\partial g_j[\boldsymbol{X}_0, t_0, \xi(\boldsymbol{X}_0, t_0, t)]}{\partial \boldsymbol{X}_0}\right)_{\text{ex}} \\ &\quad + \left(\frac{\partial g_j[\boldsymbol{X}_0, t_0, \xi(\boldsymbol{X}_0, t_0, t)]}{\partial \xi}\right)\left(\frac{\partial \xi(\boldsymbol{X}_0, t_0, t)}{\partial \boldsymbol{X}_0}\right), \\ &\quad j = 1, 2, 3, \ldots, n \end{aligned} \tag{14}$$

where the subscription "ex" indicates the derivative with respect to the initial states that appear explicitly in the function g_j.

Two-Body Relative Motion Problem

The relative motion of orbiting satellites has been a topic of great interest in recent years in connection with formation flying and proximity operations. Most often, the equations that are used to model this relative motion are "variational equations" corresponding to the two-body problem, or perhaps a perturbed two-body problem. The most popular relative motion equations are the HCW equations. As noted by several authors, including Broucke [5], Gim and Alfriend [6], Schaub and Junkins [7], and Yamanaka and Ankersen [8], the relative motion of two satellites can be described by considering the motion of each and finding the differences in their positions and velocities. We call this description the "two, two-body solution" (TTBS). Usually it is not as useful as variational equations and corresponding state transition matrices. If the assumptions regarding the magnitude of the relative position and velocity vectors made to obtain the variational equations are made with respect to the TTBS, then the TTBS may be linearized by considering the motion of one satellite to be the "reference solution" for the other. Gim and Alfriend [6] used this concept in what they termed their "geometric method," but they did not give a lot of attention to its connection to variational equations, generally. Broucke [5] used, in principal, the same method to be shown here. However, Broucke and Gim and Alfriend used orbital elements and modified orbital elements, respectively, not the initial conditions on state variables.

Considering the geometry of Fig. 1, the position of satellite S_2 is given by

$$\mathbf{R} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = [\Omega]^T[i]^T[\theta]^T \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

where

$$[\theta] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}; \qquad [i] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix}$$
$$[\Omega] = \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

Here, Ω is the longitude of the ascending node of the orbit, i is the inclination of the orbit, and R and θ are the polar coordinates of the satellite. Also shown in Fig. 1 is S_1, which we may consider to represent a satellite that we may call the "chief" satellite in a formation, or just the nominal position of the satellite of interest. The subscript N denotes a nominal, or reference, coordinate. An expansion about a non-two-body reference solution may be applied, as Gim and Alfriend [6] have done. However, here we take the reference solution to be two-body elliptic motion. We may express the position and velocity of a reference, or nominal, satellite S_1 in terms of polar coordinates R_N and θ_N, their time derivatives, and the orbital elements Ω_N and i_N as follows:

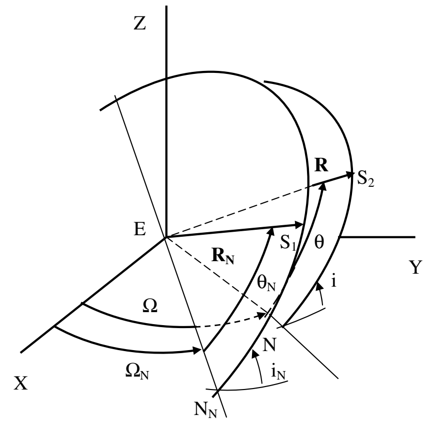

Fig. 1 Geometry of the two-satellite problem.

$$\mathbf{R}_N = \begin{bmatrix} X_N \\ Y_N \\ Z_N \end{bmatrix} = [\Omega_N]^T[i_N]^T[\theta_N]^T \begin{bmatrix} R_N \\ 0 \\ 0 \end{bmatrix} \tag{17}$$

Note that the matrices $[\Omega]$, $[i]$, and $[\theta]$ are direction cosine matrices for a 3-1-3 Euler angle sequence. (Sometimes authors, for example, Battin [9], use the transposes of the matrices given earlier and consider the transformation from the orbital system to the inertial system to be basic transformation.)

We may write nonlinear expressions for the components of the vector that describe the position of S_2 relative to its nominal position, that of satellite, or point, S_1. To do this, we use the dextral, orthogonal, "orbiting" coordinate system $Sxyz$, shown in Fig. 2. This system has its x axis along the nominal position vector $\boldsymbol{R}_N$, its y axis perpendicular to the x axis and lying in the orbital plane, and its z axis normal to the x and y axes in the right-hand sense. The y axis is the transverse axis of the cylindrical coordinate system, which has unit vectors $\hat{\boldsymbol{u}}_R$, $\hat{\boldsymbol{u}}_\theta$, and $\hat{\boldsymbol{u}}_n$. These coordinates will be referred to herein as HCW coordinates.

Using the foregoing results, we may write the elements of $\boldsymbol{r}$, the position vector of S_2 relative to S_1, as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{C}_N\mathbf{C}^T \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} R_N \\ 0 \\ 0 \end{bmatrix} \tag{18}$$

where $\mathbf{C}_N = [\theta_N][i_n][\Omega_N]$ and $\mathbf{C}^T = [\Omega]^T[i]^T[\theta]^T$. Then, the initial, relative coordinates may be obtained from

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \mathbf{C}_N(t_0)\mathbf{C}^T(t_0) \begin{bmatrix} R_0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} R_{N0} \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

Equation (18) is *exact* for relative two-body motion if we evaluate R, R_N, etc., using the appropriate solutions to the two-body problem.

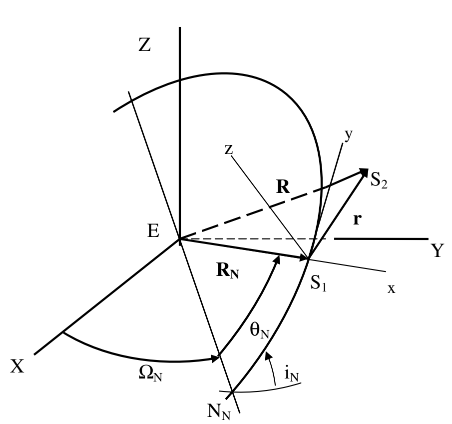

Fig. 2 Orbiting coordinate system.

The solutions to the variational equations for the two-body motion, and hence for the relative motion, may be obtained by expanding Eqs. (18) about the solution corresponding to the nominal motion. Considered another way, we may expand the right-hand side of Eqs. (18) in a first-order Taylor series to find x, y, and z, approximately, as is done with the right-hand sides of differential equations to obtain the approximate derivative functions and hence the variational equations. The difference, and a major one, is that Eqs. (18)are algebraic equations.

Let $\delta()$ denote the change in ( ) due to changes in the initial conditions R_0, θ_0, $\dot{R}_0$, $\dot{\theta}_0$, i, and Ω. Then,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cong [C_N]\delta[C]^T \begin{bmatrix} R_N \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \delta R \\ 0 \\ 0 \end{bmatrix} \tag{20}$$

In Eq. (20),

$$\delta[\mathbf{C}]^T = \mathbf{C}_N^T\tilde{\Theta} \tag{21}$$

where we have defined

$$\tilde{\Theta} \equiv \begin{bmatrix} 0 & -\Theta_z & \Theta_y \\ \Theta_z & 0 & -\Theta_x \\ -\Theta_y & \Theta_x & 0 \end{bmatrix} \tag{22}$$

In the definition for $\tilde{\Theta}$, $\Theta_x = (\delta i\cos\theta_N + \delta\Omega\sin i_N\sin\theta_N)$, $\Theta_y = (-\delta i\sin\theta_N +$
$\delta\Omega\sin i_N\cos\theta_N)$, and $\Theta_z = (\delta\theta + \delta\Omega\cos i_N)$. It follows from Eqs. (20–22) that we may write the approximate relative position equations as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & R_N & 0 & R_N\cos i_N \\ 0 & 0 & R_N\sin\theta_N & -R_N\sin i_N\cos\theta_N \end{bmatrix} \begin{bmatrix} \delta R \\ \delta\theta \\ \delta i \\ \delta\Omega \end{bmatrix} \tag{23}$$

For very small values of δi, $\delta\Omega$, Eq. (23) reduces to

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 \\ 0 & R_N & 0 \\ 0 & 0 & R_N\sin\theta_N \end{bmatrix} \begin{bmatrix} \delta R \\ \delta\theta + \theta\Omega \\ \delta i \end{bmatrix} \tag{24}$$

because when i_N is on the order of δi we can neglect second order terms and put $y \cong R_N(\delta\theta + \delta\Omega)$.

The time rate of change of Eq. (23) provides the components of the relative velocity,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \dot{R}_N & 0 & \dot{R}_N\cos i_N \\ 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}t}R_N\sin\theta_N & \frac{\mathrm{d}}{\mathrm{d}t}[-R_N\sin i_N\cos\theta_N] \end{bmatrix} \begin{bmatrix} \delta R \\ \delta\theta \\ \delta i \\ \delta\Omega \end{bmatrix} + \begin{bmatrix} \delta\dot{R} \\ R_N\delta\dot{\theta} \\ 0 \\ 0 \end{bmatrix} \tag{25}$$

Note that the time derivatives of i_N, δi, and $\delta\Omega$ are assumed to be zero because in two-body motion the orbital plane of each satellite has a constant orientation. For small i_N,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dot{R}_N & 0 \\ 0 & 0 & \dot{R}_N \sin\theta_N + R_N\dot{\theta}_N \cos\theta_N \end{bmatrix} \begin{bmatrix} \delta R \\ \delta\theta \\ \delta i \end{bmatrix} + \begin{bmatrix} \delta\dot{R} \\ R_N\delta\dot{\theta} \\ 0 \end{bmatrix} \tag{26}$$

We can write the combined Eqs. (23) and (26) as

$$\begin{bmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} \end{bmatrix}^T = \mathbf{D}\begin{bmatrix} \delta R & \delta\theta & \delta\dot{R} & \delta\dot{\theta} & \delta i & \delta\Omega \end{bmatrix}^T \tag{27}$$

where

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & R_N & 0 & 0 & 0 & R_N\cos i_N \\ 0 & 0 & 0 & 0 & R_N\sin\theta_N & -R_N\sin i_N\cos\theta_N \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \dot{R}_N & 0 & R_N & 0 & \frac{d}{dt}(R_N\cos i_N) \\ 0 & 0 & 0 & 0 & \frac{d}{dt}(R_N\sin\theta_N) & \frac{d}{dt}(-R_N\sin i_N\cos\theta_N) \end{bmatrix} \tag{28}$$

The corresponding equations for small $\mathbf{i}_N$ are

$$\begin{bmatrix} x & y & \dot{x} & \dot{y} \end{bmatrix}^T = \mathbf{D}_2\begin{bmatrix} \delta R & \delta\theta & \delta\dot{R} & \delta\dot{\theta} \end{bmatrix}^T \tag{29}$$

where

$$\mathbf{D}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & R_N & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \dot{R}_N & 0 & R_N \end{bmatrix} \tag{30}$$

and

$$\begin{bmatrix} z \\ \dot{z} \end{bmatrix} = \begin{bmatrix} R_N\sin\theta_N \\ \frac{h}{R_N}\cos\theta_N + \dot{R}_N\sin\theta_N \end{bmatrix}\delta i \tag{31}$$

where $h = R_N^2\dot{\theta}_N$ is the area velocity constant (commonly, "angular momentum") for the reference orbit.

Equations (23), (25), and (27) are the equivalent to those given by Broucke [5], but are written in terms of different variables. They are also the equivalent to those given by Gim and Alfriend [6] when the effects of the Earth's oblateness are neglected.

The relative motion normal to the nominal orbit is uncoupled from the motion in the orbital plane. The equations for z and $\dot{z}$ are

$$\begin{bmatrix} z \\ \dot{z} \end{bmatrix} = \begin{bmatrix} R_N\sin\theta_N & -R_N\sin i_N\cos\theta_N \\ \frac{h}{R_N}\cos\theta_N + \dot{R}_N\sin\theta_N & \frac{h}{R_N}\sin i_N\sin\theta_N - \dot{R}_N\sin i_N\cos\theta_N \end{bmatrix}\begin{bmatrix} \delta i \\ \delta\Omega \end{bmatrix} \tag{32}$$

At this point in the process of obtaining state transition matrices, Broucke [5] used classical orbital elements and Gim and Alfriend [6] used modified elements to write the variations of R, θ, etc. As a result, they obtain rather large matrices in their versions of the linearized form of Eqs. (18). These matrices were then inverted to get transition matrices. Here, we choose to use the initial values of the polar coordinates and their time derivatives and thereby avoid inverting any matrix except $\mathbf{D}$, which must be inverted at the initial time, to get the transition matrix. Of course, depending on the application, the inverse of the transition matrix after the latter has been obtained may be required, but this may be done (because the equations are autonomous) by just switching the variables and their initial conditions. The variables R, θ, $\dot{R}$, and $\dot{\theta}$ are determined by their initial values, and there should be no singularities, or linearly dependent equations, for the motion of S_2 relative to S_1 in the orbital plane, if we vary the initial conditions R_0, θ_0, $\dot{R}_0$, and $\dot{\theta}_0$, at $t = t_0$. We can write the initial conditions for the variations of the polar coordinates and their time derivatives in terms of the initial (approximate) relative state variables, which we will assume known or to be determined, in, for example, a guidance algorithm, as

$$\begin{bmatrix} \delta R_0 & \delta\theta_0 & \delta\dot{R}_0 & \delta\dot{\theta}_0 & \delta i & \delta\Omega \end{bmatrix}^T = [\mathbf{D}(t_0)]^{-1}\begin{bmatrix} x_0 & y_0 & z_0 & \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix}^T \tag{33}$$

where

$$\mathbf{D}^{-1}(t_0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{R_N} & \frac{\cos i_N(\dot{R}_N\sin\theta_N + R_N\cos\theta_N\dot{\theta}_N)}{R_N^2\sin i_N\dot{\theta}_N} & 0 & 0 & \frac{-\cos i_N\sin\theta_N}{R_N\sin i_N\dot{\theta}_N} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{-\dot{R}_N}{R_N^2} & 0 & 0 & \frac{1}{R_N} & 0 \\ 0 & 0 & \frac{-(\dot{R}_N\cos\theta_N - R_N\sin\theta_N\dot{\theta}_N)}{R_N^2\dot{\theta}_N} & 0 & 0 & \frac{\cos\theta_N}{R_N\dot{\theta}_N} \\ 0 & 0 & \frac{-(\dot{R}_N\sin\theta_N + R_N\cos\theta_N\dot{\theta}_N)}{R_N^2\sin i_N\dot{\theta}_N} & 0 & 0 & \frac{\sin\theta_N}{R_N\sin i_N\dot{\theta}_N} \end{bmatrix}_{t=t_0} \tag{34}$$

Notice that, as discussed earlier, $\sin i_N$ appears in the denominator of some terms of the inverse. Therefore, the alternate equations, Eqs. (24) and (26), should be used if the inclination of the reference orbit is very small.

If we put

$$\begin{bmatrix} \delta R & \delta\theta & \delta\dot{R} & \delta\dot{\theta} & \delta i & \delta\Omega \end{bmatrix}^T = \mathbf{B}(t, t_0)\begin{bmatrix} \delta R_0 & \delta\theta_0 & \delta\dot{R}_0 & \delta\dot{\theta}_0 & \delta i & \delta\Omega \end{bmatrix}^T \tag{35}$$

then Eqs. (35) may be rewritten as

$$\begin{bmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} \end{bmatrix}^T = \mathbf{\Phi}(t, t_0)\begin{bmatrix} x_0 & y_0 & z_0 & \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix}^T \tag{36}$$

where $\mathbf{\Phi}(t, t_0) = \mathbf{DBD}^{-1}(t_0)$.

Now, all that remains is to find $\mathbf{B}$ whose partial derivatives are only a first-order Taylor series approximation. We may write $\mathbf{B}$ as

$$\mathbf{B} = \begin{bmatrix} \frac{\partial R}{\partial R_0} & \frac{\partial R}{\partial \theta_0} & \frac{\partial R}{\partial \dot{R}_0} & \frac{\partial R}{\partial \dot{\theta}_0} & 0 & 0 \\ \frac{\partial \theta}{\partial R_0} & \frac{\partial \theta}{\partial \theta_0} & \frac{\partial \theta}{\partial \dot{R}_0} & \frac{\partial \theta}{\partial \dot{\theta}_0} & 0 & 0 \\ \frac{\partial \dot{R}}{\partial R_0} & \frac{\partial \dot{R}}{\partial \theta_0} & \frac{\partial \dot{R}}{\partial \dot{R}_0} & \frac{\partial \dot{R}}{\partial \dot{\theta}_0} & 0 & 0 \\ \frac{\partial \dot{\theta}}{\partial R_0} & \frac{\partial \dot{\theta}}{\partial \theta_0} & \frac{\partial \dot{\theta}}{\partial \dot{R}_0} & \frac{\partial \dot{\theta}}{\partial \dot{\theta}_0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{37}$$

As is often the case in astrodynamics, a lot of the work required to find the matrix $\mathbf{B}$ was done some time ago, by Battin [9] and others. Although Battin does not give the exact forms of the results we need here, he does give "most of the pieces of the puzzle." The first step is to express R, θ, $\dot{R}$, and $\dot{\theta}$ in terms of R_0, θ_0, $\dot{R}_0$, and $\dot{\theta}_0$. To that end, we write the integrals of the motion in terms of R_0, θ_0, $\dot{R}_0$, and $\dot{\theta}_0$. The energy equation is

$$\frac{1}{2}\left[\dot{R}_{00}^2 + R_0^2\dot{\theta}_0^2\right] - \frac{\mu}{R_0} = -\frac{\mu}{2a} \tag{38}$$

where μ is the gravitational parameter and a is the semimajor axis of the reference orbit, so that

$$a = \mu R_0\left[2\mu - R_0\dot{R}_0^2 - R_0^3\dot{\theta}_0^2\right]^{-1} \tag{39}$$

Also, the areal velocity (angular momentum) integral is $h = R_0^2\dot{\theta}_0$ so that the eccentricity is $e = [1 - R_0^4\dot{\theta}_0^2/(\mu a)]^{1/2}$. But, we do not have to use the eccentricity in our solution. Instead (see [9], p. 130), we use the fact that $f - f_0 = \theta - \theta_0$, where f is the true anomaly, and that $f + \omega = \theta = \theta - \theta_0 + \theta_0$ to write the orbit equation, $R=$

$p/(1+e\cos f)$, where $p=a(1-e)^2$ is the semilatus rectum of the reference orbit. Because $\theta=f+\omega$ and $f=\theta-\omega$, $R=\frac{p}{1+e\cos f}=\frac{p}{1+e\cos\omega\cos\theta+e\sin\omega\sin\theta}$ and $\dot{R}=\frac{eh}{p}\sin f=\frac{eh}{p}[\sin\theta\cos\omega-\cos\theta\sin\omega]$. Evaluating these expressions at $t=t_0$, we get $R_0=\frac{p}{1+e\cos\omega\cos\theta_0+e\sin\omega\sin\theta_0}$ and $\dot{R}_0=\frac{eh}{p}[\sin\theta_0\cos\omega-\cos\theta_0\sin\omega]$. These last two equations may be rewritten in the matrix form

$$\begin{bmatrix}\cos\theta_0 & \sin\theta_0\\ \sin\theta_0 & -\cos\theta_0\end{bmatrix}\begin{bmatrix}e\cos\omega\\ e\sin\omega\end{bmatrix}=\begin{bmatrix}\frac{p}{R_0}-1\\ \frac{p}{h}\dot{R}_0\end{bmatrix}$$

Thus, we have

$$e\cos\omega=\left(\frac{p}{R_0}-1\right)\cos\theta_0+\left(\frac{p\dot{R}_0}{h}\right)\sin\theta_0$$

and

$$e\sin\omega=\left(\frac{p}{R_0}-1\right)\sin\theta_0-\left(\frac{p\dot{R}_0}{h}\right)\cos\theta_0$$

so that

$$\begin{aligned}e\cos\omega\cos\theta+e\sin\omega\sin\theta&=\left[\left(\frac{p}{R_0}-1\right)\cos\theta_0\right.\\&\quad\left.+\left(\frac{p\dot{R}_0}{h}\right)\sin\theta_0\right]\cos\theta+\left[\left(\frac{p}{R_0}-1\right)\sin\theta_0\right.\\&\quad\left.-\left(\frac{p\dot{R}_0}{h}\right)\cos\theta_0\right]\sin\theta=\left[\left(\frac{p}{R_0}-1\right)\cos(\theta-\theta_0)\right.\\&\quad\left.-\left(\frac{p\dot{R}_0}{h}\right)\sin(\theta-\theta_0)\right]\end{aligned}$$

Because $p=\frac{h^2}{\mu}$ and $h=R_0^2\dot{\theta}$, we finally get

$$R=\frac{R_0^4\dot{\theta}_0^2/\mu}{1+\left(R_0^3\dot{\theta}_0^2/\mu-1\right)\cos(\theta-\theta_0)-\left(R_0^2\dot{\theta}_0\dot{R}_0/\mu\right)\sin(\theta-\theta_0)}\tag{40}$$

Similarly, we can write $\dot{R}=(eh/p)\sin f$ as

$$\dot{R}=\frac{eh}{p}\sin f=\frac{eh}{p}\sin(\theta-\omega)=\frac{h}{p}[e\sin\theta\cos\omega-e\cos\theta\sin\omega]\tag{41}$$

where

$$\begin{aligned}\frac{h}{p}[e\sin\theta\cos\omega-e\cos\theta\sin\omega]&=\frac{h}{p}\left[\left(\frac{p}{R_0}-1\right)\cos\theta_0\right.\\&\quad\left.+\left(\frac{p\dot{R}_0}{h}\right)\sin\theta_0\right]\sin\theta-\frac{h}{p}\left[\left(\frac{p}{R_0}-1\right)\sin\theta_0\right.\\&\quad\left.-\left(\frac{p\dot{R}_0}{h}\right)\cos\theta_0\right]\cos\theta=\frac{h}{p}\left[\left(\frac{p}{R_0}-1\right)\sin(\theta-\theta_0)\right.\\&\quad\left.-\left(\frac{p\dot{R}_0}{h}\right)\cos(\theta-\theta_0)\right]\end{aligned}$$

Hence,

$$\dot{R}=\dot{R}_0\cos(\theta-\theta_0)+\left[R_0\dot{\theta}_0-\mu/\left(R_0^2\dot{\theta}_0\right)\right]\sin(\theta-\theta_0)\tag{42}$$

Furthermore, the equation for the angular momentum can be used along with Eqs. (40) for R to get

$$\begin{aligned}\dot{\theta}&=\frac{\mu^2}{R_0^6\dot{\theta}_0^3}\left[1+\left(R_0^3\dot{\theta}_0^2/\mu-1\right)\cos(\theta-\theta_0)\right.\\&\quad\left.-\left(R_0^2\dot{\theta}_0\dot{R}_0/\mu\right)\sin(\theta-\theta_0)\right]^2\end{aligned}\tag{43}$$

Finding the partial derivatives of R, $\dot{R}$, and $\dot{\theta}$, with respect to the initial conditions is straightforward, but tedious. Our results for the partial derivatives are given in the Appendix. Getting a similar equation for θ and its derivatives is more complicated.

As noted earlier, here, we consider only circular and elliptic orbits. For such orbits, we know that Kepler's equation relates the eccentric anomaly and the time. We also know that the true anomaly, and hence the orbital longitude, may be determined from the eccentric anomaly. Thus, we can use Battin's Eq. (4.43) ([9], p. 164),

$$\begin{aligned}M-M_0&=E-E_0+\frac{R_0\dot{R}_0}{\mu a}[1-\cos(E-E_0)]\\&\quad-(1-R_0/a)\sin(E-E_0)\end{aligned}\tag{44}$$

and write it as

$$n(t-t_0)=v+K_1(1-\cos v)+K_2\sin v\tag{45}$$

where $n=\sqrt{\mu/a^3}$, $v=E-E_0$, $K_1=R_0\dot{R}_0/\sqrt{\mu a}$, and $K_2=R_0/a-1$. Equation (45) is our time equation.

From the different forms for the F and G functions ([9], pp. 130 and 162) in terms of v and $u=\theta-\theta_0$, we can get the following identities relating u and v:

$$\cos u=1-\left(R_0^3\dot{\theta}_0^2/\mu\right)(a/R)(1-\cos v)\tag{46a}$$

$$\sin u=(a/R)\left[\left(R_0^2\dot{\theta}_0\dot{R}_0/\mu\right)(1-\cos v)+(R_0\dot{\theta}_0/\sqrt{\mu a})\sin v\right]\tag{46b}$$

Derivatives of Eqs. (46) with respect to the initial conditions may be obtained and then combined to get derivatives of u, in terms of the derivatives of v. These are derived from Eqs. (45). Note that by considering Eqs. (4.42) of [9], we can write

$$\frac{\partial}{\partial()}[n(t-t_0)]=\frac{R}{a}\frac{\partial v}{\partial()}+\frac{\partial K_1}{\partial()}\sin v+\frac{\partial K_2}{\partial()}\cos v\tag{47}$$

Some guidance as to how many partial derivatives we need to calculate can be found by considering the two-body variational equations,

$$\begin{bmatrix}\delta\dot{R}\\ \delta\dot{\theta}\\ \delta\ddot{R}\\ \delta\ddot{\theta}\end{bmatrix}=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \frac{2\mu}{R_N^3}+\dot{\theta}_N^2 & 0 & 0 & R_N\dot{\theta}_N\\ \frac{2\dot{R}_N\dot{\theta}_N}{R_N^2} & 0 & \frac{-2\dot{\theta}}{R_N} & \frac{-2\dot{R}_N}{R_N}\end{bmatrix}\begin{bmatrix}\delta R\\ \delta\theta\\ \delta\dot{R}\\ \delta\dot{\theta}\end{bmatrix}\tag{48}$$

Note that the $\delta\theta$ column contains all zeroes because θ is an ignorable coordinate. This means that the only nonzero partial with respect θ_0 is $\partial\theta/\partial\theta_0=1$. Thus, we need three derivatives each for δR and $\delta\theta$.

To find the solutions for $\delta\dot{R}$ and $\delta\dot{\theta}$, we can take the time derivatives of the solutions for δR and $\delta\theta$. However, because $P_\theta=R^2\dot{\theta}=\text{constant}$,

$$\delta P_\theta=2\delta R\dot{\theta}_N+R_N^2\delta\dot{\theta}=2\delta R_0\dot{\theta}_0+R_0^2\delta\dot{\theta}_0=0\tag{49}$$

Thus, we have the solution,

$$\delta\dot{\theta}=\frac{2\dot{\theta}_0}{R_N^2}\delta R_0+\frac{R_0^2}{R_N^2}\delta\dot{\theta}_0-\frac{2\dot{\theta}_N}{R_N^2}\delta R\tag{50}$$

for $\delta\dot{\theta}$ once we have found the solution for δR. Also, we have the energy integral to find $\delta\dot{R}$. However, it appears to be easier to find the partial derivatives of $\dot{R}$. Again, these derivatives are given in the Appendix.

Relationship to HCW Transition Matrix

So far, we have developed the transition matrix for arbitrary elliptic orbits. This matrix must have certain properties. One is that it is an identity matrix at the initial time t_0, that is, $\mathbf{\Phi}(t_0,t_0)=$

Table 1 Simulation parameters

Parameter	Value
Eccentricity	$e = 0.1$ and 0.4
Perigee height	500 km
Inclination	$i = 45$ deg
Longitude of ascending node	$\Omega = 100$ deg
Argument of periapsis	$\omega = 120$ deg
True anomaly	$v = 0$ deg
Chaser initial relative position in the CW coordinate	[1 1 1] km
Chaser initial relative velocity in the CW coordinate	$10^{-3} * [1\ \ 1\ \ 1]$ km/s
Propagation time	Two orbit periods, one half-orbit period
Numerical integration method	Runge–Kutta seventh order, 1 or 0.1 s step size

$\mathbf{D}(t_0)\mathbf{B}(t_0)\mathbf{D}^{-1}(t_0) = \mathbf{I}_{6\times 6}$. This has been verified. Also, when the eccentricity of the reference orbit is zero, the Cochran–Lee–Jo (CLJ) [10] transition matrix and the HCW transition matrix should be equal. This is difficult to verify analytically, but with $e_N = 0$ we used the CLJ and HCW transition matrices to calculate numerically the magnitudes of the differences in the relative position and velocity vectors for a large number of cases. For two orbital periods, the numerical values of the magnitudes of the position difference were on the order of 10^{-11} m and those for the magnitudes of the velocity differences were on the order of 10^{-14} m/s. The data for an example are given in Table 1 and magnitude results are shown in Figs. 3 and 4, respectively.

Verification of Φ

The solution to the variational equations represented by the CLJ transition matrix contains many partial derivatives. These have been checked by the authors several times, some by using the symbolic algebra toolbox in MATLAB™. One check on the algebra consists of verifying that the matrix is an identity at $t = t_0$, which was done. Also, the elements of the matrix Φ may be approximated numerically. To obtain the numerical results that follow, the orbital elements of the target satellite are assumed to be known. These elements are used to compute the position and velocity of the target satellite (or nominal position and velocity) in inertial coordinates and to obtain initial conditions for the two-body equations in the polar coordinates R and θ. Then, the two-body equations in polar coordinates are integrated numerically to obtain the target satellite, or nominal motion. When the initial relative position and velocity in the HCW coordinates are given, incremental state components δR_0, $\delta\theta_0$, $\delta\dot{R}_0$, $\delta\dot{\theta}_0$ are calculated using Eqs. (33). The step size is an important factor affecting the accuracy in the numerical integration. Thus, to

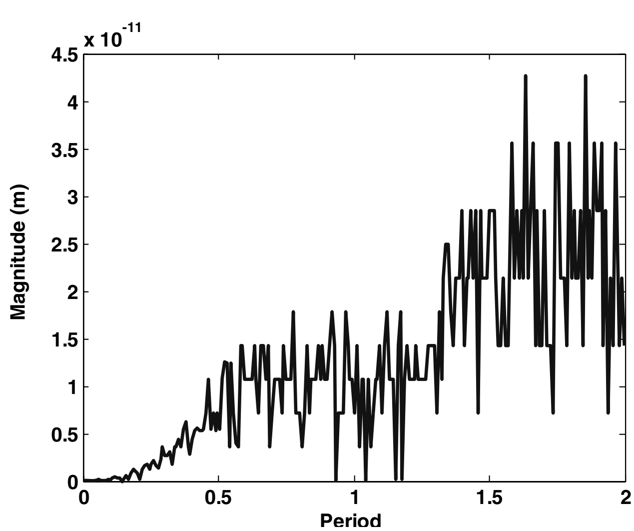

Fig. 3 Magnitude of position vector difference.

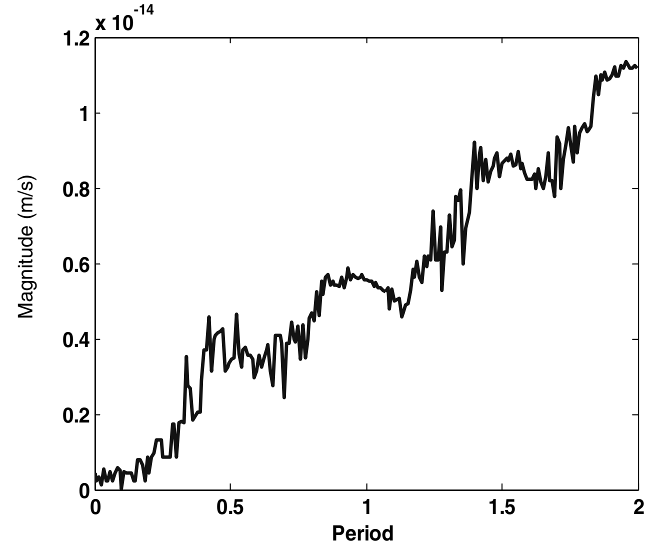

Fig. 4 Magnitude of velocity difference.

achieve better accuracy in the numerical integrations, a seventh-order Runge–Kutta algorithm was used with a 0.1 s step size instead of 1 s. The local truncation error in this algorithm is $\mathcal{O}(h^6)$ [11,12], where h is the step size.

The core portion of Φ is the four-by-four upper left-hand partition of the $\mathbf{B}$ matrix. This is the solution to the two-body variational equations in terms of the variations of the polar coordinates of the in-plane motion. Numerical values of the elements of $\mathbf{B}$, obtained by evaluating the analytical expressions for the elements, may be compared with numerical results that are obtained by "varying the initial conditions" as indicated in Fig. 5. The procedure, although computationally expensive, is algorithmically simple. As an example, to calculate $\mathbf{B}(1,1) = \partial R/\partial R_0$, for the half period of interest, the changes in R from the nominal solution that occur when only R_0 is varied are found. Also, to find $\mathbf{B}(4,1) = \partial\dot{\theta}/\partial R_0$, the changes in $\dot{\theta}$ due to a small change in R_0 are calculated over the half period of interest. The results for the first column of $\mathbf{B}$ for the $e = 0.4$ case described in Table 1 are shown in Figs. 6–9. The results agree in both value and first derivative at the initial time and for about one half

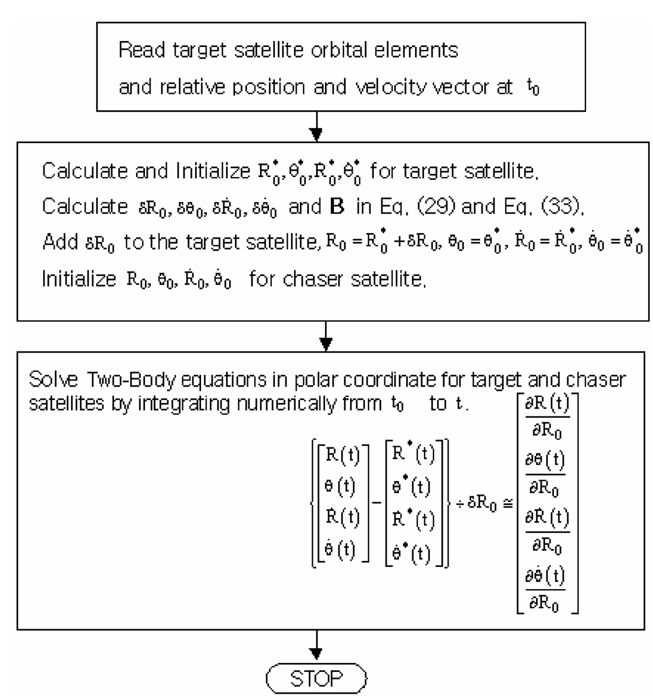

Fig. 5 Process for numerical calculation of the elements of B.

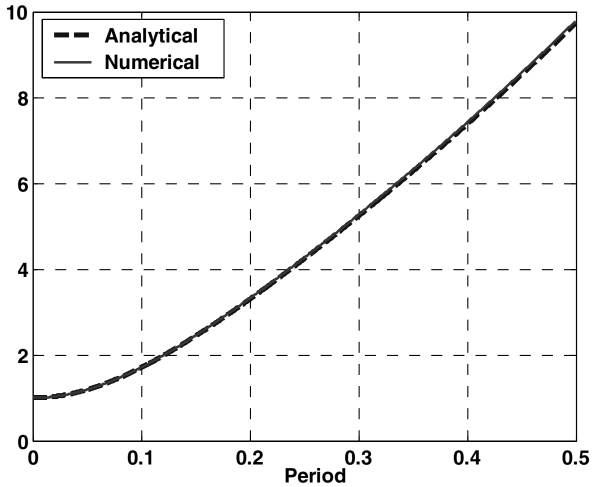

Fig. 6 Element $\mathbf{B}(1,1) = \frac{\partial R}{\partial R_0}$.

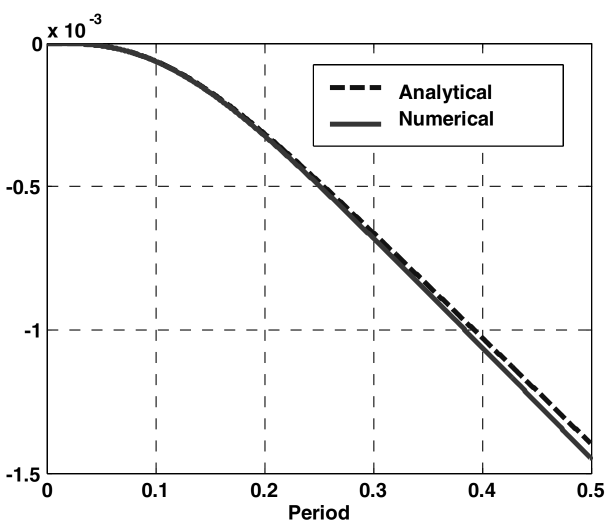

Fig. 7 Partial derivative $\mathbf{B}(2,1) = \frac{\partial \theta}{\partial R_0}$.

period. Afterwards, the errors incurred in the numerical process build up fairly rapidly.

We have also checked the calculation of the relative position and velocity by numerically integrating the linearized relative motion equations for the case of an elliptic reference orbit [see, for example,

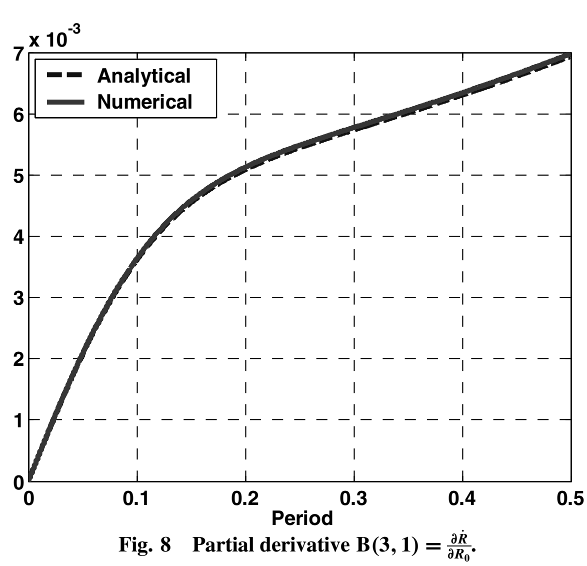

Fig. 8 Partial derivative $\mathbf{B}(3,1) = \frac{\partial \dot{R}}{\partial R_0}$.

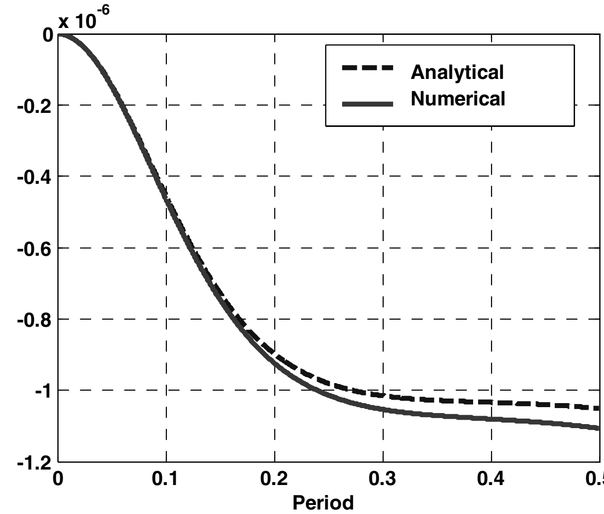

Fig. 9 Partial derivative $\mathbf{B}(4,1) = \frac{\partial \dot{\theta}}{\partial R_0}$.

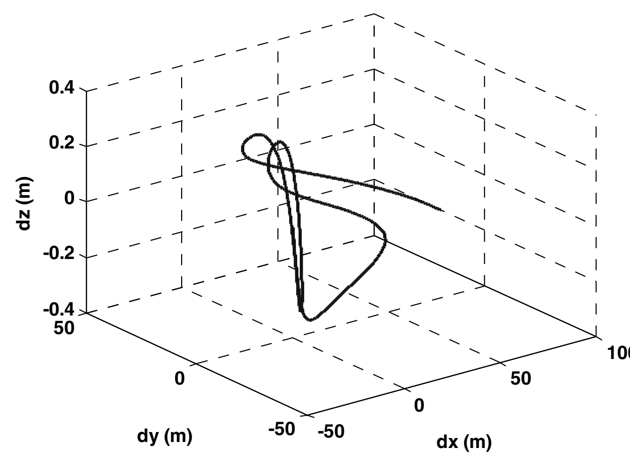

Fig. 10 Difference in relative motion solutions.

[7], p. 599, Eqs. (14.18)]. The results are extremely close, as they should be. The difference in the solutions for two periods is shown in Fig. 10. Numerical experiments indicate that the use of a smaller step size in the numerical integration reduces the difference in the solutions. This indicates that the difference is due to the numerical integration. The magnitudes of the differences in relative position and velocity from the transition matrix solution and the numerical solution of the linear equations are shown in Figs. 11 and 12, respectively.

Additional verification of the correctness of the CLJ transition matrix can be obtained by comparing linear solutions with solutions obtained using the TTB formulation for cases in which $|\delta \mathbf{R}(t_0)| \ll |\mathbf{R}_N(t_0)|$. This has been done for numerous cases with excellent results. Here, we consider an example case of target and chaser spacecraft in the Earth-centered inertial coordinate system. The results, for the case described in Table 1 with $e = 0.1$, are given in Figs. 13 and 14, respectively. It can be seen from these results that the solution from CLJ transition matrix is, as expected, quite accurate, whereas the HCW solutions are not.

Conclusions

The derivation of analytical solutions to the variational equations of completely integrable nonlinear, autonomous, dynamical systems has been discussed. Because such equations are often needed in the development of guidance and control algorithms, analytical solutions to the variational equations for "unperturbed" problems, like the two-body problem, are very useful. An analytical solution for the state transition matrix for the linearized relative motion of two

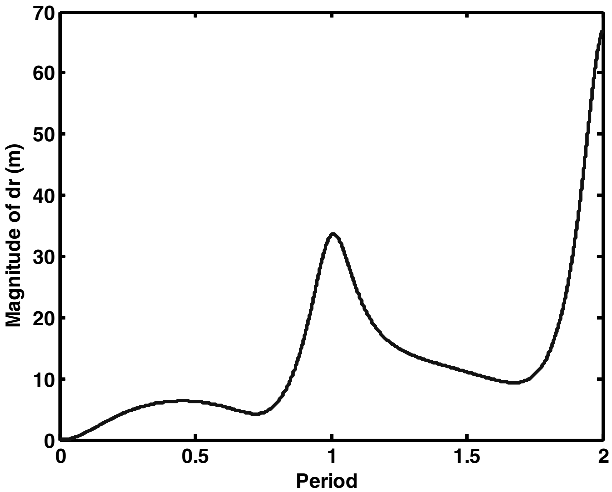

Fig. 11 Magnitude of the difference in position vector.

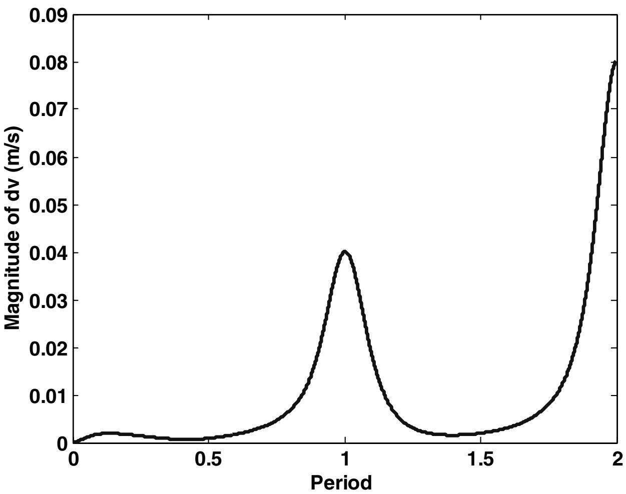

Fig. 12 Magnitude of the difference in relative velocity.

satellites, valid for eccentricities less than one, was obtained. This solution is similar to others, but the choice of variables appears to be unique. The core portion of the relative motion state transition matrix is a four-by-four state transition matrix that is the solution to the two-body variational equations in terms of the variations of the polar coordinates of the in-plane motion. Its initial evaluation is not as complicated as transition matrices based or other variables. Several methods have been used to verify that the elements of the new transition matrix have been derived correctly. Numerical experiments indicate that the linear solution is an accurate representation of the small perturbation relative motion with respect to circular and elliptic reference orbits.

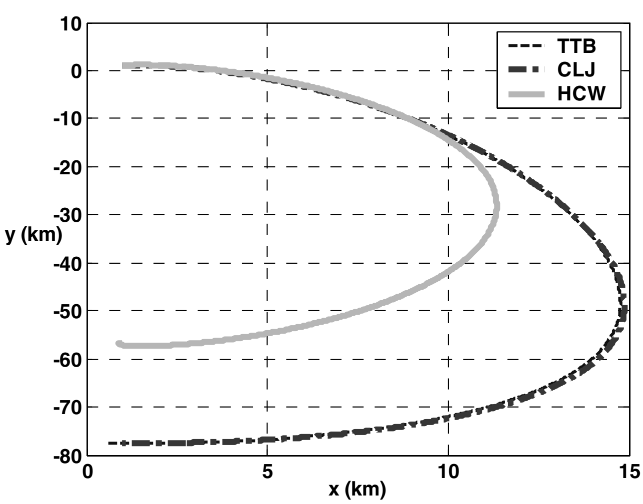

Fig. 13 Comparison of solutions in the x–y plane for $e = 0.1$.

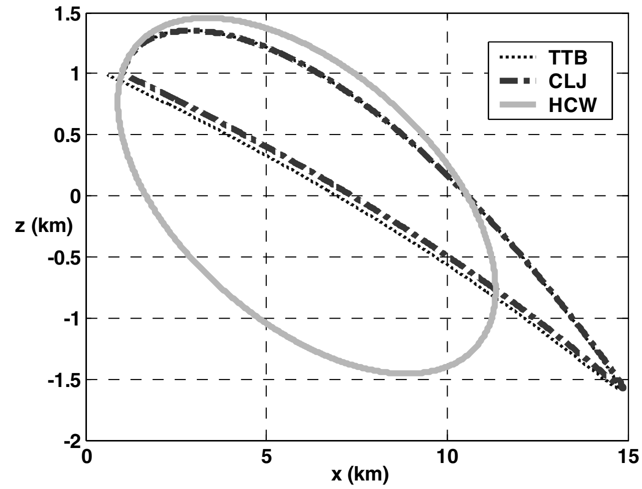

Fig. 14 Comparison of solutions in the x–z plane for $e = 0.1$.

Appendix: Partial Derivatives

Partial Derivatives of R

$$\frac{\partial R}{\partial R_0} = \left(\frac{\partial R}{\partial R_0}\right)_{\text{ex}} + \left(\frac{\partial R}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial R_0}\right)$$

where

$$\left(\frac{\partial R}{\partial \theta}\right) = -\left(\frac{\partial R}{\partial \theta_0}\right)_{\text{ex}}$$

and subscript "ex" denotes explicit derivative.

$$\frac{\partial R}{\partial R_0} = 4\frac{R}{R_0} - 3\frac{R^2}{R_0^2}\cos(\theta-\theta_0) + \frac{2R^2\dot{R}_0}{R_0^3\dot{\theta}_0}\sin(\theta-\theta_0) + \frac{R^2}{p}\left[\left(\frac{p}{R_0}-1\right)\sin(\theta-\theta_0) + \frac{p\dot{R}_0}{h}\cos(\theta-\theta_0)\right]\left(\frac{\partial \theta}{\partial R_0}\right)$$

$$\frac{\partial R}{\partial \theta_0} = \left(\frac{\partial R}{\partial \theta_0}\right)_{\text{ex}} + \left(\frac{\partial R}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial \theta_0}\right) = \left(\frac{\partial R}{\partial \theta_0}\right)_{\text{ex}} - \left(\frac{\partial R}{\partial \theta}\right)_{\text{ex}}\left(\frac{\partial \theta}{\partial \theta_0}\right) = \left(\frac{\partial R}{\partial \theta_0}\right)_{\text{ex}} - \left(\frac{\partial R}{\partial \theta}\right)_{\text{ex}}(1) = 0$$

$$\frac{\partial R}{\partial \dot{R}_0} = \left(\frac{\partial R}{\partial \dot{R}_0}\right)_{\text{ex}} + \left(\frac{\partial R}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial \dot{R}_0}\right) = \left(\frac{\partial R}{\partial \dot{R}_0}\right)_{\text{ex}} - \left(\frac{\partial R}{\partial \theta_0}\right)_{\text{ex}}\left(\frac{\partial \theta}{\partial \dot{R}_0}\right)$$

$$\frac{\partial R}{\partial \dot{R}_0} = \frac{R^2}{h}\sin(\theta-\theta_0) + \frac{R^2}{p}\left[\left(\frac{p}{R_0}-1\right)\sin(\theta-\theta_0) + \frac{p\dot{R}_0}{h}\cos(\theta-\theta_0)\right]\left(\frac{\partial \theta}{\partial \dot{R}_0}\right)$$

$$\frac{\partial R}{\partial \dot{\theta}_0} = \left(\frac{\partial R}{\partial \dot{\theta}_0}\right)_{\text{ex}} + \left(\frac{\partial R}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial \dot{\theta}_0}\right) = \left(\frac{\partial R}{\partial \dot{\theta}}\right)_{\text{ex}} - \left(\frac{\partial R}{\partial \theta_0}\right)_{\text{ex}}\left(\frac{\partial \theta}{\partial \dot{\theta}_0}\right)$$

$$\frac{\partial R}{\partial \dot{\theta}_0} = 2\frac{R}{\dot{\theta}_0} - \frac{2R^2}{R_0\dot{\theta}_0}\cos(\theta-\theta_0) + \frac{R^2\dot{R}_0}{\dot{\theta}^2 R_0^2}\sin(\theta-\theta_0) + \frac{R^2}{p}\left[\left(\frac{p}{R_0}-1\right)\sin(\theta-\theta_0) + \frac{p\dot{R}_0}{h}\cos(\theta-\theta_0)\right]\left(\frac{\partial\theta}{\partial\dot{\theta}_0}\right)$$

Partial Derivatives of θ

$$\frac{\partial\theta}{\partial R_0} = \frac{-\sin u(f_{R1}) + \cos u(f_{R2})}{1+\sin u(C_1/R)(1-\cos v)(\partial R/\partial\theta) + \cos u[(S_1/R)(\partial R/\partial\theta)(1-\cos v) + (S_2/R)(\partial R/\partial\theta)(\sin \mathrm{v})]}$$

where $u = \theta - \theta_0$, $v = E - E_0$,

$$C_1 = \frac{R_0^3\dot{\theta}_0^2}{\mu}\left(\frac{a}{R}\right)$$

$$S_1 = \left(\frac{R_0^2\dot{\theta}_0}{R}\right)\left(\frac{a}{\mu}\right)\dot{R}_0, \qquad S_2 = \frac{R_0^2\dot{\theta}_0}{R}\sqrt{\frac{a}{\mu}}$$

$$f_{R1} = \left(\frac{C_1}{R}\right)\left(\frac{\partial R}{\partial R_0}\right)_{\mathrm{ex}}(1-\cos v) - \left(\frac{C_1}{a}\right)\left(\frac{\partial a}{\partial R_0}\right)(1-\cos v) - C_1\sin v\left(\frac{\partial v}{\partial R0}\right)$$

$$f_{R2} = \left\{2\left(\frac{S_1}{R_0}\right) - \left(\frac{S_1}{R}\right)\left(\frac{\partial R}{\partial R_0}\right)_{\mathrm{ex}} + \left(\frac{S_1}{a}\right)\left(\frac{\partial a}{\partial R_0}\right)\right\}(1-\cos v) + \left\{\left(\frac{2S_2}{R_0}\right) - \left(\frac{S_2}{R}\right)\left(\frac{\partial R}{\partial R_0}\right)_{\mathrm{ex}} + \frac{1}{2}\left(\frac{S_2}{a}\right)\left(\frac{\partial a}{\partial R_0}\right)\right\}\sin v + [S_1\sin v + S_2\cos v]\left(\frac{\partial v}{\partial R_0}\right)$$

$$\left(\frac{\partial v}{\partial R_0}\right) = \left(\frac{a}{R}\right)\left[\frac{\partial n}{\partial R_0}(t-t_0) - \left(\frac{\partial K_1}{\partial R_0}\right)(1-\cos v) - \left(\frac{\partial K_2}{\partial}\right)\sin v\right]$$

$$\frac{\partial n}{\partial R_0} = -\frac{3}{2}\left(\frac{n}{a}\right)\left(\frac{\partial a}{\partial R_0}\right)$$

$$K_1 = \frac{\dot{R}_0 R_0}{\sqrt{\mu a}}, \qquad K_2 - \frac{R_0}{a} - 1, \qquad n = \sqrt{\frac{\mu}{a^3}}$$

$$\left(\frac{\partial K_1}{\partial R_0}\right) = \frac{\dot{R}_0}{\sqrt{\mu a}} - \frac{1}{2}\frac{K_1}{a}\left(\frac{\partial a}{\partial R_0}\right), \qquad \left(\frac{\partial K_2}{\partial R_0}\right) = \frac{1}{a} - \frac{R_0}{a}\frac{\partial a}{\partial R_0}$$

$$\frac{\partial\theta}{\partial\theta_0} = 1$$

$$\frac{\partial\theta}{\partial\dot{R}_0} = \frac{-\sin u(f_{\dot{R}1}) + \cos u(f_{\dot{R}2})}{1+\sin u(C_1/R)(1-\cos v)(\partial R/\partial\theta) + \cos u[(S_1/R)(\partial R/\partial\theta)(1-\cos v) + (S_2/R)(\partial R/\partial\theta)(\sin v)]}$$

$$f_{\dot{R}1} = \left[-\sin u - \left(\frac{C_1}{R}\right)\left(\frac{\partial R}{\partial\theta}\right)(1-\cos v)\right]\left(\frac{\partial\theta}{\partial\dot{R}_0}\right) \quad = \left(\frac{C_1}{R}\right)\left(\frac{\partial R}{\partial\dot{R}_0}\right)_{\mathrm{ex}}(1-\cos v) - \left(\frac{C_1}{a}\right)\left(\frac{\partial a}{\partial\dot{R}_0}\right)(1-\cos v) - C_1\sin v\frac{\partial v}{\partial\dot{R}_0}$$

$$f_{\dot{R}2} = \left[\cos u + \left(\frac{S_1}{R}\right)\left(\frac{\partial R}{\partial\theta}\right)(1-\cos v) + \left(\frac{S_2}{R}\right)\left(\frac{\partial R}{\partial\theta}\right)\sin v\right]\frac{\partial\theta}{\partial\dot{R}_0} = \left[\left(\frac{R_0^2\dot{\theta}_0}{R}\right)\left(\frac{a}{\mu}\right) - \left(\frac{S_1}{R}\right)\left(\frac{\partial R}{\partial\dot{R}_0}\right)_{\mathrm{ex}} + \left(\frac{S_1}{a}\right)\left(\frac{\partial a}{\partial\dot{R}_0}\right)\right](1-\cos v) + \left[-\left(\frac{S_2}{R}\right)\left(\frac{\partial R}{\partial\dot{R}_0}\right)_{\mathrm{ex}} + \left(\frac{S_2}{a}\right)\frac{\partial a}{\partial\dot{R}_0}\right]\sin v + [S_1\sin v + S_2\cos v]\left(\frac{\partial v}{\partial\dot{R}_0}\right)$$

Partial Derivatives of $\dot{R}$

$$\frac{\partial \dot{R}}{\partial R_0} = \left(\frac{\partial \dot{R}}{\partial R_0}\right)_{\text{ex}} + \left(\frac{\partial \dot{R}_0}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial R_0}\right)$$

$$\frac{\partial \dot{R}}{\partial R_0} = \left(\dot{\theta}_0 + 2\frac{\mu}{R_0^3\dot{\theta}_0}\right)\sin(\theta-\theta_0) - \left[\dot{R}_0\sin(\theta-\theta_0) - \left(R_0\dot{\theta}_0 - \frac{\mu}{R_0^2\dot{\theta}_0}\right)\cos(\theta-\theta_0)\right]\left(\frac{\partial \theta}{\partial R_0}\right)$$

$$\frac{\partial \dot{R}}{\partial \theta_0} = \left(\frac{\partial \dot{R}_0}{\partial \theta_0}\right)_{\text{ex}} + \left(\frac{\partial \dot{R}_0}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial \theta_0}\right) = \left(\frac{\partial \dot{R}_0}{\partial \theta_0}\right)_{\text{ex}} - \left(\frac{\partial \dot{R}_0}{\partial \theta_0}\right)_{\text{ex}}\left(\frac{\partial \theta}{\partial \theta_0}\right) = \left(\frac{\partial \dot{R}_0}{\partial \theta_0}\right)_{\text{ex}} - \left(\frac{\partial \dot{R}_0}{\partial \theta_0}\right)_{\text{ex}}(1) = 0$$

$$\frac{\partial \dot{R}}{\partial \dot{R}_0} = \left(\frac{\partial \dot{R}}{\partial \dot{R}_0}\right)_{\text{ex}} + \left(\frac{\partial \dot{R}}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial \dot{R}_0}\right) = \cos(\theta-\theta_0) - \left(\frac{\partial \dot{R}}{\partial \theta_0}\right)_{\text{ex}}\left(\frac{\partial \theta}{\partial \dot{R}_0}\right) = \cos(\theta-\theta_0) - \left\{\dot{R}_0\sin(\theta-\theta_0) - \left(\dot{R}_0\dot{\theta}_0 - \frac{\mu}{R_0^2\dot{\theta}_0}\right)\sin(\theta-\theta_0)\right\}\left(\frac{\partial \theta}{\partial \dot{R}_0}\right)$$

$$\frac{\partial \dot{R}}{\partial \dot{\theta}_0} = \left(\frac{\partial \dot{R}}{\partial \dot{\theta}_0}\right)_{\text{ex}} + \left(\frac{\partial \dot{R}}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial \dot{\theta}_0}\right) = \left(\frac{\partial \dot{R}}{\partial \dot{\theta}_0}\right)_{\text{ex}} - \left(\frac{\partial \dot{R}}{\partial \theta_0}\right)_{\text{ex}}\left(\frac{\partial \theta}{\partial \dot{\theta}_0}\right)$$

$$\frac{\partial \dot{R}}{\partial \dot{\theta}_0} = \left(R_0 + \frac{\mu}{R_0^2\dot{\theta}_0^2}\right)\sin(\theta-\theta_0) - \left[\dot{R}_0\sin(\theta-\theta_0) - \left(\dot{R}_0\dot{\theta}_0 - \frac{\mu}{R_0^2\dot{\theta}_0}\right)\sin(\theta-\theta_0)\right]\left(\frac{\partial \theta}{\partial \dot{\theta}_0}\right)$$

Partial Derivatives of $\dot{\theta}$

$$\frac{\partial \dot{\theta}}{\partial R_0} = \left(\frac{\partial \dot{\theta}}{\partial R_0}\right)_e + \left(\frac{\partial \dot{\theta}}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial R_0}\right) = \left(\frac{\partial \dot{\theta}}{\partial R_0}\right)_e - \left(\frac{\partial \dot{\theta}}{\partial \theta_0}\right)_e\left(\frac{\partial \theta}{\partial R_0}\right)$$

$$\frac{\partial \dot{\theta}}{\partial R_0} = \left(-6\frac{\mu^2}{R_0^7\dot{\theta}_0^3}\right)x^2 + \left(\frac{\mu^2}{R_0^6\dot{\theta}_0^3}\right)(2x)\left[\left(3\frac{R_0^2\dot{\theta}_0^2}{\mu}\right)\cos(\theta-\theta_0) - 2\frac{R_0\dot{R}_0\dot{\theta}_0}{\mu}\sin(\theta-\theta_0)\right] - \left\{\left(\frac{\mu^2}{R_0^6\dot{\theta}_0^3}\right)(2x)\left[\left(\frac{R_0^3\dot{\theta}_0^2}{\mu} - 1\right)\sin(\theta-\theta_0) + \frac{R_0^2\dot{R}_0\dot{\theta}_0}{\mu}\cos(\theta-\theta_0)\right]\right\}_e\left(\frac{\partial \theta}{\partial R_0}\right)$$

where

$$x = \left[1 + \left(\frac{R_0^3\dot{\theta}_0^2}{\mu} - 1\right)\cos(\theta-\theta_0) - \frac{R_0^2\dot{R}_0\dot{\theta}_0}{\mu}\sin(\theta-\theta_0)\right]$$

$$\frac{\partial \dot{\theta}}{\partial \theta_0} = \left(\frac{\partial \dot{\theta}}{\partial \theta_0}\right)_{\text{ex}} + \left(\frac{\partial \dot{\theta}}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial \theta_0}\right) = \left(\frac{\partial \dot{\theta}}{\partial \theta_0}\right)_{\text{ex}} - \left(\frac{\partial \dot{\theta}}{\partial \theta_0}\right)_{\text{ex}}\left(\frac{\partial \theta}{\partial \theta_0}\right) = \left(\frac{\partial \dot{\theta}}{\partial \theta_0}\right)_{\text{ex}} - \left(\frac{\partial \dot{\theta}}{\partial \theta_0}\right)_{\text{ex}}(1) = 0$$

$$\frac{\partial \dot{\theta}}{\partial \dot{R}_0} = \left(\frac{\partial \dot{\theta}}{\partial \dot{R}_0}\right)_e + \left(\frac{\partial \dot{\theta}}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial \dot{R}_0}\right) = \left(\frac{\partial \dot{\theta}}{\partial \dot{R}_0}\right)_e - \left(\frac{\partial \dot{\theta}}{\partial \theta_0}\right)_e\left(\frac{\partial \theta}{\partial \dot{R}_0}\right)$$

$$\frac{\partial \dot{\theta}}{\partial \dot{R}_0} = \left(\frac{\mu^2}{R_0^6\dot{\theta}_0^3}\right)(2x)\left(-\frac{R_0^2\dot{\theta}_0}{\mu}\sin(\theta-\theta_0)\right) - \left\{\left(\frac{\mu^2}{R_0^6\dot{\theta}_0^3}\right)(2x)\left[\left(\frac{R_0^3\dot{\theta}_0^2}{\mu} - 1\right)\sin(\theta-\theta_0) + \frac{R_0^2\dot{R}_0\dot{\theta}_0}{\mu}\sin(\theta-\theta_0)\right]\right\}\left(\frac{\partial \theta}{\partial \dot{R}_0}\right)$$

$$\frac{\partial \dot{\theta}}{\partial \dot{\theta}_0} = \left(\frac{\partial \dot{\theta}}{\partial \theta_0}\right)_e + \left(\frac{\partial \dot{\theta}}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial \dot{\theta}_0}\right) = \left(\frac{\partial \dot{\theta}}{\partial \theta_0}\right)_e - \left(\frac{\partial \dot{\theta}}{\partial \theta_0}\right)_e\left(\frac{\partial \theta}{\partial \dot{\theta}_0}\right)$$

$$\frac{\partial \dot{\theta}}{\partial \dot{\theta}_0} = \left\{\left(-3\frac{\mu^2}{R_0^6\dot{\theta}_0^4}\right)x^2 + \left(\frac{\mu^2}{R_0^6\dot{\theta}_0^3}\right)(2x)\left[\left(2\frac{R_0^3\dot{\theta}_0}{\mu}\right)\cos(\theta-\theta_0) + \frac{R_0^2\dot{R}_0}{\mu}\sin(\theta-\theta_0)\right]\right\} - \left\{\left(\frac{\mu^2}{R_0^6\dot{\theta}_0^3}\right)(2x)\left[\left(\frac{R_0^3\dot{\theta}_0^2}{\mu} - 1\right)\sin(\theta-\theta_0) + \frac{R_0^2\dot{R}_0\dot{\theta}_0}{\mu}\sin(\theta-\theta_0)\right]\right\}\left(\frac{\partial \theta}{\partial \dot{\theta}_0}\right)$$

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